A LEGENDRE TRANSFORM ON AN EXOTIC S^3

VITTORIO MARTINO

Abstract. We consider an exotic contact form α on S^3 and we establish explicitly the existence of a non singular vector field v in $ker(\alpha)$ such that the non-singular one-differential form $\beta(\cdot) := d\alpha(v, \cdot)$ is a contact form on S^3 with the same orientation than α. In particular this means that a Legendre transform can be completed.

1. Introduction

In this paper we consider an exotic contact form α on S^3, introduced by J.Gonzalo-F.Varela in ([4], case $n = 1$). It is, according to ([4]) an overtwisted contact structure and in Appendix A we can actually find an explicit disk D^2 whose boundary is a Legendrian curve for α and $ker(\alpha)$ has exactly one point of tangency to D^2. This contact structure is therefore not standard. The standard contact form α_0 on S^3 is a pull-back from the standard contact form on $P(\mathbb{R}^3)$, that is the unit sphere cotangent bundle of S^2; therefore it is equipped with its Liouville form. Legendre duality can be completed for this Liouville form. This Legendre transform can be viewed as the data of a vector field v in $ker(\alpha_0)$ such that $\beta_0(\cdot) := d\alpha_0(v, \cdot)$ is a contact form with the same orientation than α_0.

This Legendre transform allows the transformation of a Hamiltonian problem on the cotangent sphere of S^2 into a Lagrangian problem. This duality has been extended by A.Bahri-D.Bennequin in ([1]) to the more general framework of a contact form α on a three-dimensional compact orientable manifold without boundary M, leading

Dipartimento di Matematica, Università di Bologna, Piazza di Porta S.Donato 5, 40126 Bologna, Italy. E-mail address: martino@dm.unibo.it.

to a variational problem on a spaces of curves. In fact if one assumes that:

(i) $\exists\, v \in TM$, a non-vanishing vector field, such that $v \in ker(\alpha)$

(ii) the non-singular one-differential form $\beta(\cdot) := d\alpha(v, \cdot)$ is a contact form on M with the same orientation than α

by defining the action functional

$$J(x) = \int_0^1 \alpha(\dot{x})dt \tag{1}$$

on the subspace of the H^1-loops on M:

$$C_\beta = \{x \in H^1(S^1; M)\ s.t.\ \beta(\dot{x}) = 0;\ \alpha(\dot{x}) = \text{strictly positive constant}\}$$

If $\xi \in TM$ denotes the Reeb vector field of α, i.e.

$$\alpha(\xi) = 1, \qquad d\alpha(\xi, \cdot) = 0 \tag{2}$$

then the following result by A.Bahri-D.Bennequin holds ([1]):

Theorem 1.1. *J is a C^2 functional on C_β whose critical points are periodic orbits of ξ.*

It is important to observe that this construction is "stable under perturbation", that is the same v can be used to complete Legendre duality for forms $\lambda\alpha$, with $\lambda \in C^2$ and $|\lambda - 1|$ small.

In this work we establish the existence of such a v, which is given explicitly, for the contact structure of J.Gonzalo-F.Varela.

The organization of the paper is the following: in Section 2 we verify the hypothesis (i) giving explicitly the vector field v; in Section 3 we verify the hypothesis (ii); we conclude the paper with four appendices. In Appendix A, we provide an explicit disk that allows to recognize a known fact about the contact structure of α, namely that it is overtwisted. Appendix B is devoted to the graphs of some of the (complicated) functions that we use. Our v is C^∞ outside of two curves. It is only C^0 on S^3. We regularize it (with a very standard and straightforward regularizing procedure; v is in fact C^∞ in the direction of the Reeb vector field ξ) in Appendix C so that it is now C^∞ and hypotheses (i) and (ii) are still satisfied. We then study in Appendix D the case $n > 1$ of the contact forms/structures of Gonzalo-Varela ([4]). The definition of v extends, but hypothesis (ii) is not satisfied anymore by this extension. Another extension might work.

Acknowledgement I was introduced to this topic by Professor Abbas Bahri during the year I was visiting him at Rutgers University first and Courant Institute then, so it's a pleasure to thank him for all his help, support and valuable hints.

2. Verification of hypothesis (i)

From now on we consider S^3 as embedded submanifold of $\mathbb{R}^4$ where we will carry on most of our computation. Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, by denoting

$$r_1 = x_1^2 + x_2^2, \qquad r_2 = x_3^2 + x_4^2$$

then

$$S^3 = \{x \in \mathbb{R}^4 : r_1 + r_2 = 1\}$$

and on S^3 we take the non-standard (or exotic) contact form α defined by J.Gonzalo and F.Varela in ([4], case $n = 1$):

$$\alpha = -\Big(A(x_2dx_1 - x_1dx_2) + B(x_4dx_3 - x_3dx_4)\Big)$$

where

$$\theta = \frac{\pi}{4} + \pi r_2, \qquad A = \cos\theta, \qquad B = \sin\theta$$

Now we compute $d\alpha$. If we denote by

$$\widetilde{A} = A + \pi r_1 B = \frac{\partial}{\partial r_1}(r_1 A) \qquad \widetilde{B} = B + \pi r_2 A = \frac{\partial}{\partial r_2}(r_2 B)$$

then by a direct computation

$$d\alpha = 2\Big(\widetilde{A}dx_1 \wedge dx_2 + \widetilde{B}dx_3 \wedge dx_4\Big) \tag{3}$$

Now, if

$$\zeta = -\Big(\widetilde{B}(x_2\partial_{x_1} - x_1\partial_{x_2}) + \widetilde{A}(x_4\partial_{x_3} - x_3\partial_{x_4})\Big)$$

one has $\zeta \in T(S^3)$ and it holds [1]

$$\alpha(\zeta) = A\widetilde{B}r_1 + B\widetilde{A}r_2 > 0, \qquad d\alpha(\zeta, \cdot) = 0 \tag{4}$$

Thus the Reeb vector field of α is

$$\xi = \frac{\zeta}{\alpha(\zeta)} \tag{5}$$

Let us define the following non singular[2] vector field in $T(S^3)$

$$T = -\Big(A(x_2\partial_{x_1} - x_1\partial_{x_2}) + B(x_4\partial_{x_3} - x_3\partial_{x_4})\Big) \tag{6}$$

so one finds

$$\alpha(\cdot) = < T, \cdot > \tag{7}$$

where $< \cdot, \cdot >$ is the usual inner product in $\mathbb{R}^4$. In other words, a vector field is in the kernel of α if it is orthogonal to T.

Theorem 2.1. *Let $R := |T|$, where*

$$|T|^2 = < T, T > = \alpha(T) = A^2 r_1 + B^2 r_2 > 0$$

By letting $C = A/R$ and $D = B/R$ let us define the vector field

$$v = v_1\partial_{x_1} + v_2\partial_{x_2} + v_3\partial_{x_3} + v_4\partial_{x_4}$$

[1] See Appendix B

[2] See Appendix B

with

$$
\begin{cases}
v_1 = x_3 \dfrac{(x_1^2 - Dx_2^2)}{r_1} + \dfrac{(x_1 x_2 x_4)}{r_1}(1+D) \\
v_2 = x_4 \dfrac{(x_2^2 - Dx_1^2)}{r_1} + \dfrac{(x_1 x_2 x_3)}{r_1}(1+D) \\
v_3 = -x_1 \dfrac{(x_3^2 + Cx_4^2)}{r_2} - \dfrac{(x_2 x_3 x_4)}{r_2}(1-C) \\
v_4 = -x_2 \dfrac{(x_4^2 + Cx_3^2)}{r_2} - \dfrac{(x_1 x_3 x_4)}{r_2}(1-C)
\end{cases} \tag{8}
$$

Then $v \in T(S^3)$, $|v| = 1$ and $v \in ker(\alpha)$, so the condition (i) is satisfied.

Proof. We introduce the two objects

$$M = S^3 \setminus (\{r_1 = 0\} \cup \{r_2 = 0\}) \tag{9}$$

and

$$T^2 = \{r_1 = c_1, r_2 = c_2, c_1 + c_2 = 1, c_1 \neq 0, c_2 \neq 0\} \tag{10}$$

So T^2 are invariant tori for ξ (i.e $\xi \in T(T^2)$) and M is the sphere without the two degenerate tori (circles). Moreover, also the vector field T is tangent to T^2. We introduce the following two vector fields in $T(M)$

$$X = \frac{1}{\sqrt{r_1 r_2}} \Big(D r_2 (x_2 \partial_{x_1} - x_1 \partial_{x_2}) - C r_1 (x_4 \partial_{x_3} - x_3 \partial_{x_4}) \Big) \tag{11}$$

$$Y = \frac{1}{\sqrt{r_1 r_2}} \Big(r_2 (x_1 \partial_{x_1} + x_2 \partial_{x_2}) - r_1 (x_3 \partial_{x_3} + x_4 \partial_{x_4}) \Big) \tag{12}$$

It holds

$$|X| = |Y| = 1$$

thus X, Y are non degenerate on M. Moreover $X, Y \in ker(\alpha)$, in particular $X \in T(T^2)$ and $Y \in N(T^2)$ (the normal space to T^2). With the following coefficients

$$a = \frac{1}{\sqrt{r_1 r_2}}(x_1 x_3 + x_2 x_4), \qquad b = \frac{1}{\sqrt{r_1 r_2}}(x_1 x_4 - x_2 x_3), \qquad a^2 + b^2 = 1 \tag{13}$$

let us define

$$v = aY + bX \tag{14}$$

So $v \in ker(\alpha)$, $|v| = 1$ and by a direct computation one finds the coefficients in (8). Let us remark that v is defined only on M. Since

$$\lim_{r_2 \to 0} C = - \lim_{r_1 \to 0} D = 1$$

on $r_1 = 0$ one has

$$v = x_3 \partial_{x_1} + x_4 \partial_{x_2}$$

whereas on $r_2 = 0$ one finds

$$v = -x_1 \partial_{x_3} - x_2 \partial_{x_4}$$

so, by continuity, v is defined on the whole S^3. □

Corollary 2.1. *In the same way if we define the vector field*

$$w = w_1 \partial_{x_1} + w_2 \partial_{x_2} + w_3 \partial_{x_3} + w_4 \partial_{x_4}$$

with

$$\begin{cases} w_1 = -x_4 \dfrac{(x_1^2 - D x_2^2)}{r_1} + \dfrac{(x_1 x_2 x_3)}{r_1}(1 + D) \\ \\ w_2 = x_3 \dfrac{(x_2^2 - D x_1^2)}{r_1} - \dfrac{(x_1 x_2 x_4)}{r_1}(1 + D) \\ \\ w_3 = -x_2 \dfrac{(x_3^2 + C x_4^2)}{r_2} + \dfrac{(x_1 x_3 x_4)}{r_2}(1 - C) \\ \\ w_4 = x_1 \dfrac{(x_4^2 + C x_3^2)}{r_2} - \dfrac{(x_2 x_3 x_4)}{r_2}(1 - C) \end{cases} \tag{15}$$

Then $w \in T(S^3)$, $|w| = 1$, $w \in ker(\alpha)$ and $w \perp v$.

Proof. The proof is the same as in (2.1), with

$$w = aX - bY \tag{16}$$

So $w \perp v$, $w \in ker(\alpha)$, $|w| = 1$. Moreover on $r_1 = 0$ one has

$$w = -x_4\partial_{x_1} + x_3\partial_{x_2}$$

whereas on $r_2 = 0$ one finds

$$w = -x_2\partial_{x_3} + x_1\partial_{x_4}$$

□

Remark 2.1. *We want to point out that the (coefficients of the) vector fields v, w are by construction only C^0.*

3. Verification of hypothesis (ii)

Let us consider now the non-singular one-differential form

$$\beta(\cdot) := d\alpha(v, \cdot) \tag{17}$$

By defining $h := \alpha(\zeta)$, one has

$$d\alpha(v,w) = d\alpha(aY + bX, aX - bY) = (a^2 + b^2)d\alpha(Y,X) = d\alpha(Y,X) = -\frac{2}{|T|}h < 0$$

and

$$\alpha \wedge d\alpha(\zeta, v, w) = hd\alpha(v,w) < 0$$

Moreover[3]

$$\beta \wedge d\beta(\zeta, v, w) = \beta(w)d\beta(\zeta, v) = -d\alpha(v,w)d\alpha(v, [\zeta, v])$$

Thus

$$\frac{\beta \wedge d\beta(\zeta, v, w)}{\alpha \wedge d\alpha(\zeta, v, w)} = \frac{-d\alpha(v, [\zeta, v])}{h} \tag{18}$$

Theorem 3.1. $d\alpha(v, [\zeta, v]) < 0$*, so the condition (ii) is satisfied.*

[3] The vector field v is C^0 so in order to compute $[\zeta, v]$ we need to regularize v, see Appendix C

Proof. We explicitly write some formulas. For $0 < r_1 < 1$, one finds

$$Y(r_1) = 2\sqrt{r_1 r_2}, \qquad Y(r_2) = -2\sqrt{r_1 r_2}, \qquad Y(\theta) = -2\pi\sqrt{r_1 r_2}$$

$$Y(A) = 2\pi\sqrt{r_1 r_2}B, \qquad Y(B) = -2\pi\sqrt{r_1 r_2}A$$

$$Y(\widetilde{A}) = 2\pi\sqrt{r_1 r_2}(2B - \pi r_1 A), \qquad Y(\widetilde{B}) = -2\pi\sqrt{r_1 r_2}(2A - \pi r_2 B)$$

$$\zeta(a) = -(\widetilde{A} - \widetilde{B})b, \qquad \zeta(b) = (\widetilde{A} - \widetilde{B})a$$

$$[\zeta, X] = 0$$

$$[\zeta, Y] = Y(\widetilde{B})(x_2\partial_{x_1} - x_1\partial_{x_2}) + Y(\widetilde{A})(x_4\partial_{x_3} - x_3\partial_{x_4})$$

Moreover

$$[\zeta, v] = [\zeta, aY + bX] = \zeta(a)Y + \zeta(b)X + a[\zeta, Y] + b[\zeta, X] =$$

$$= (\widetilde{A} - \widetilde{B})w + a[\zeta, Y] = (\widetilde{A} - \widetilde{B})w + a\{Y(\widetilde{B})(x_2\partial_{x_1} - x_1\partial_{x_2}) + Y(\widetilde{A})(x_4\partial_{x_3} - x_3\partial_{x_4})\} =$$

$$= (\widetilde{A} - \widetilde{B})w + 2\pi(x_1x_3 + x_2x_4)\{-(2A - \pi r_2 B)(x_2\partial_{x_1} - x_1\partial_{x_2}) + (2B - \pi r_1 A)(x_4\partial_{x_3} - x_3\partial_{x_4})\}$$

and

$$\lim_{r_1\to 0}[\zeta, v] = \frac{\pi}{\sqrt{2}}(-x_4\partial_{x_1} + x_3\partial_{x_2}), \qquad \lim_{r_2\to 0}[\zeta, v] = \frac{\pi}{\sqrt{2}}(-x_2\partial_{x_3} + x_1\partial_{x_4}) \tag{19}$$

By computing

$$d\alpha(v, [\zeta, v]) = d\alpha(v, (\widetilde{A} - \widetilde{B})w + a[\zeta, Y]) =$$

$$= -2(\widetilde{A} - \widetilde{B})\frac{h}{R} + a d\alpha(aY + bX, \{Y(\widetilde{B})(x_2\partial_{x_1} - x_1\partial_{x_2}) + Y(\widetilde{A})(x_4\partial_{x_3} - x_3\partial_{x_4})\})$$

and by letting

$$K := \widetilde{A}(\pi r_2 B - 2A) + \widetilde{B}(\pi r_1 A - 2B)$$

one has

$$d\alpha(v, [\zeta, v]) = -2\Big\{(\widetilde{A} - \widetilde{B})\frac{h}{R} + 2\pi a^2 r_1 r_2 K\Big\} =: -2Q$$

and[4] $Q > 0$. □

[4]See Appendix B

4. Appendix A

Let us consider on S^3 the following disk

$$D^2 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ s.t. \ x_1^2 + x_2^2 \leq \frac{3}{4}, \ x_4 \geq 0, \ x_3 = \varepsilon\}$$

with $0 < \varepsilon \ll 1$.

Then the boundary of D^2 is a Legendrian curve for the contact form α (i.e. a curve in the kernel of the contact form), in fact

$$\partial D^2 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ s.t. \ x_1^2 + x_2^2 = \frac{3}{4}, \ x_4 \geq 0, \ x_3 = \varepsilon\}$$

thus $\theta|_{\partial D^2} = \frac{\pi}{2}$, $A|_{\partial D^2} = 0$ and $\alpha(\partial D^2) = 0$. Now let us consider the identically zero form on S^3

$$\omega = x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + x_4 dx_4$$

that on D^2 it reads as $x_1 dx_1 + x_2 dx_2 + x_4 dx_4$. To find the points of tangency between $ker(\alpha)$ and D^2 we can see whether $\omega = \lambda\alpha$ for some non zero real λ. Then it should be

$$\begin{cases} Ax_1 = \lambda x_2 \\ Ax_2 = -\lambda x_2 \\ B\varepsilon = \lambda x_4 \end{cases} \tag{20}$$

that means in particular

$$A(x_1^2 + x_2^2) = 0$$

Now if $A = 0$ then $\lambda = 0$, thus the only possible case is when $x_1^2 + x_2^2 = 0$ and the only one point of tangency between $ker(\alpha)$ and D^2 is $(0, 0, \varepsilon, \sqrt{1 - \varepsilon^2})$.

5. Appendix B

In this section we show the behavior (in particular the non-negativity) of some functions we used before. From now on let us put $x := r_2$.

First we study $h : [0, 1] \to \mathbb{R}$,

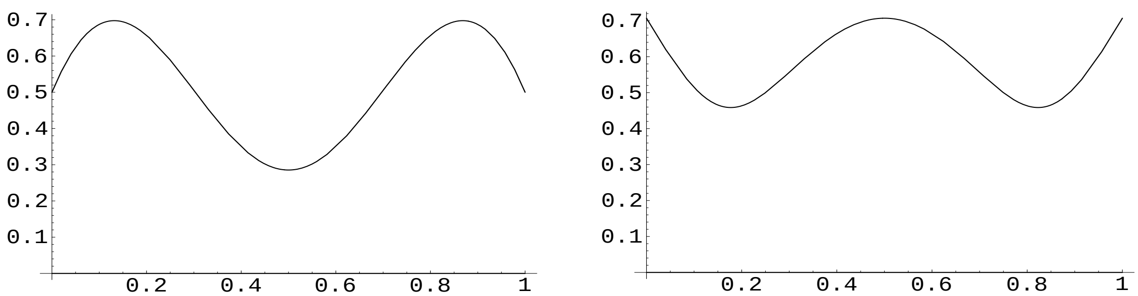
$$h(x) := \alpha(\zeta(x)) = A(x)\widetilde{B}(x)(1 - x) + B(x)\widetilde{A}(x)x = \frac{\sin(2\theta(x))}{2} + \pi(x - x^2)$$

where $\theta(x) = \pi(\frac{1}{4} + x)$. Since h is symmetric respect to $x = 1/2$, we can consider it only for $x \in [0, 1/2]$. So:

$$h'(x) = \pi(\cos(2\theta(x)) + 1 - 2x)$$

$$h''(x) = -2\pi(\pi \sin(2\theta(x)) + 1) = -2\pi(\pi \cos(2\pi x) + 1)$$

thus there exists c_1, with $1/4 < c_1 < 1/2$ such that h'' is positive definite on $(c_1, 1/2)$ and h' is increasing on $(c_1, 1/2)$. Moreover $h'(1/2) = 0$. Thus there exists c_2, with $0 < c_2 < c_1 < 1/2$ such that $h'(c_2) = 0$ and h is increasing on $(0, c_2)$. Finally, since $h(0) = 1/2$, the minimum of h is $h(1/2) = -1/2 + \pi/4 > 0$ One has also



(a) h (b) R

$R(x) := |T(x)| > 0$. Indeed

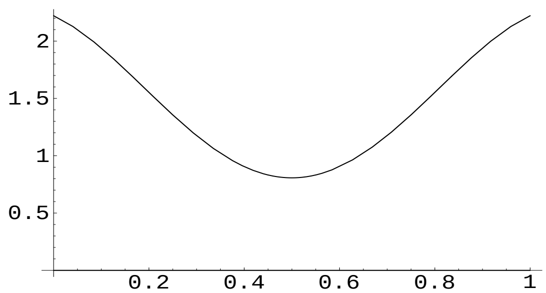
$$R^2(x) = |T(x)|^2 =< T(x), T(x) >= \alpha(T(x)) = A^2(x)(1-x) + B^2(x)x \geq 0$$

and the quantities $A^2(x)(1-x)$ and $B^2(x)x$ cannot be zero simultaneously. We prove now that $Q(x) = (\widetilde{A} - \widetilde{B})\frac{h}{R} + 2\pi a^2 r_1 r_2 K > 0$ for $x \in [0, 1]$ showing the graphs of some function (a straightforward computation is possible, as for the function h, R, to localize critical points). If

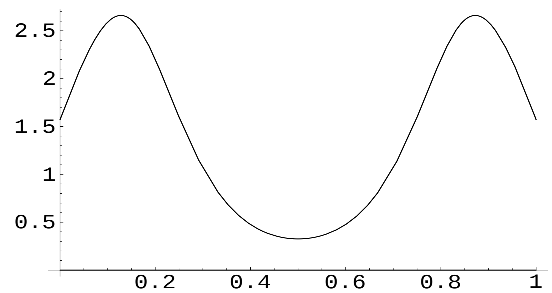
$$H(x) := (\widetilde{A}(x) - \widetilde{B}(x))\frac{h(x)}{R(x)}$$

Where K is positive definite then $Q > 0$.
Otherwise, if we define

$$G := 2\pi r_1 r_2, \qquad G(x) = 2\pi(x - x^2)$$



(c) $\widetilde{A} - \widetilde{B}$



(d) H

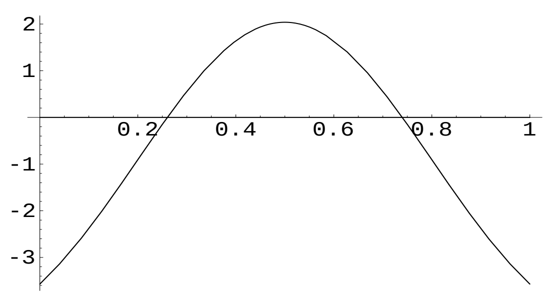
since $a^2 \leq 1$, where K is negative definite

$$2\pi a^2 r_1 r_2 K \geq GK$$

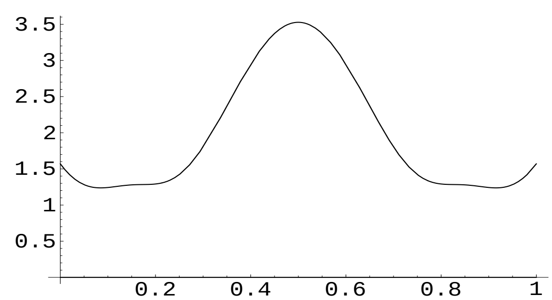
Thus, where K is negative definite, it holds

$$Q(x) \geq H(x) + G(x)K(x) =: W(x)$$

and $Q > 0$, for every $x \in [0, 1]$.



(e) K



(f) W

6. Appendix C

In this section we want to regularize the C^0 vector field v. The only problems are on the circles $r_1 = 0$ and $r_2 = 0$, otherwise v is a C^∞ vector field. Let us consider the case $r_1 = 0$, the other one is similar. Let $\mathcal{U}$ be a tubular neighborhood of $r_1 = 0$ and consider on $\mathcal{U}$ a basis $\{v_1, v_2\}$ of $ker(\alpha)$ with $v_1, v_2 \in C^\infty$ (it is not difficult to find a local C^∞ vector field). Then

$$v = a_1 v_1 + a_2 v_2 \tag{21}$$

with $a_1, a_2 \in C^0$. After a convolution with a quite standard mollifiers we can find $a_1^\varepsilon, a_2^\varepsilon \in C^\infty$ on $\mathcal{U}$, with $\varepsilon > 0$. Thus we define the C^∞ vector field

$$v^\varepsilon = a_1^\varepsilon v_1 + a_2^\varepsilon v_2 \tag{22}$$

Now, by formulas (19) we already know that the vector field $[\zeta, v] \in C^0$, then using

$$|\zeta(a_1^\varepsilon) - \zeta(a_1)| = o(1), \quad |\zeta(a_2^\varepsilon) - \zeta(a_2)| = o(1), \quad as \quad \varepsilon \to 0$$

we have on $\mathcal{U}$

$$[\zeta, v] = \lim_{\varepsilon \to 0} [\zeta, v^\varepsilon]$$

Thus, in order to compute $\beta \wedge d\beta$, we can use that

$$d\alpha(v, [\zeta, v]) = \lim_{\varepsilon \to 0} d\alpha(v^\varepsilon, [\zeta, v^\varepsilon])$$

7. Appendix D

The exotic contact form α we considered on S^3, actually is the first one of a family of non standard contact forms introduced by J.Gonzalo-F.Varela in ([4]). In fact, for every integer $n \geq 1$ let us define

$$\theta_n = \frac{\pi}{4} + n\pi r_2, \qquad A_n = \cos\theta_n, \qquad B_n = \sin\theta_n$$

$$\widetilde{A}_n = A_n + n\pi r_1 B_n = \frac{\partial}{\partial r_1}(r_1 A_n) \qquad \widetilde{B}_n = B_n + n\pi r_2 A_n = \frac{\partial}{\partial r_2}(r_2 B_n)$$

then ([4])

Theorem 7.1 (J.Gonzalo - F.Varela)**.** *The non-singular one-differential forms*

$$\alpha_n = -\Big(A_n(x_2 dx_1 - x_1 dx_2) + B_n(x_4 dx_3 - x_3 dx_4)\Big)$$

are non-standard contact forms on S^3, for every $n \geq 1$

If

$$\zeta_n = -\Big(\widetilde{B}_n(x_2\partial_{x_1} - x_1\partial_{x_2}) + \widetilde{A}_n(x_4\partial_{x_3} - x_3\partial_{x_4})\Big)$$

one has

$$h_n := \alpha_n(\zeta_n) = A_n\widetilde{B}_n r_1 + B_n\widetilde{A}_n r_2 > 0, \qquad d\alpha_n(\zeta_n, \cdot) = 0$$

and then the Reeb vector field of α_n is

$$\xi_n = \frac{\zeta_n}{h_n}$$

Now, using the same arguments as in Section 2 we can see that if

$$T_n = -\Big(A_n(x_2\partial_{x_1} - x_1\partial_{x_2}) + B_n(x_4\partial_{x_3} - x_3\partial_{x_4})\Big)$$

then one finds $|T_n| > 0$ and it holds

Theorem 7.2. *Let us define* $C_n = A_n/|T_n|, D_n = B_n/|T_n|$.
Let $n \geq 1$ *be odd.*
If $v_n = v_n^1\partial_{x_1} + v_n^2\partial_{x_2} + v_n^3\partial_{x_3} + v_n^4\partial_{x_4}$, *with*

$$\begin{cases} v_n^1 = x_3\dfrac{(x_1^2 - D_n x_2^2)}{r_1} + \dfrac{(x_1x_2x_4)}{r_1}(1 + D_n) \\ \\ v_n^2 = x_4\dfrac{(x_2^2 - D_n x_1^2)}{r_1} + \dfrac{(x_1x_2x_3)}{r_1}(1 + D_n) \\ \\ v_n^3 = -x_1\dfrac{(x_3^2 + C_n x_4^2)}{r_2} - \dfrac{(x_2x_3x_4)}{r_2}(1 - C_n) \\ \\ v_n^4 = -x_2\dfrac{(x_4^2 + C_n x_3^2)}{r_2} - \dfrac{(x_1x_3x_4)}{r_2}(1 - C_n) \end{cases}$$

Then v_n *is a non singular* C^0 *vector field in* $ker(\alpha_n)$ *and* $|v_n| = 1$.
Let $n \geq 1$ *be even.*

If $v_n = v_n^1\partial_{x_1} + v_n^2\partial_{x_2} + v_n^3\partial_{x_3} + v_n^4\partial_{x_4}$, *with*

$$\begin{cases} v_n^1 = x_3\dfrac{(x_1^2 + D_n x_2^2)}{r_1} - \dfrac{(x_1x_2x_4)}{r_1}(1 - D_n) \\ \\ v_n^2 = -x_4\dfrac{(x_2^2 + D_n x_1^2)}{r_1} + \dfrac{(x_1x_2x_3)}{r_1}(1 - D_n) \\ \\ v_n^3 = -x_1\dfrac{(x_3^2 + C_n x_4^2)}{r_2} + \dfrac{(x_2x_3x_4)}{r_2}(1 - C_n) \\ \\ v_n^4 = x_2\dfrac{(x_4^2 + C_n x_3^2)}{r_2} - \dfrac{(x_1x_3x_4)}{r_2}(1 - C_n) \end{cases}$$

Then v_n *is a non singular* C^0 *vector field in* $ker(\alpha_n)$ *and* $|v_n| = 1$.

Thus the hypothesis (i) holds by using the previous v_n.
Putting $\beta_n(\cdot) = d\alpha_n(v_n, \cdot)$ then in order to compute $\beta_n \wedge d\beta_n$ we need to know the sign of $d\alpha_n(v_n, [\zeta_n, v_n])$. By a direct computation:
if $n \geq 1$ is odd then

$$d\alpha_n(v_n, [\zeta_n, v_n]) = -2\Big\{(\widetilde{A}_n - \widetilde{B}_n)\frac{h_n}{R_n} + 2n\pi a^2 r_1 r_2 K_n\Big\} =: -2Q_n$$

where

$$K_n := \widetilde{A}_n(n\pi r_2 B_n - 2A_n) + \widetilde{B}_n(n\pi r_1 A_n - 2B_n), \qquad a = \frac{1}{\sqrt{r_1 r_2}}(x_1x_3 + x_2x_4)$$

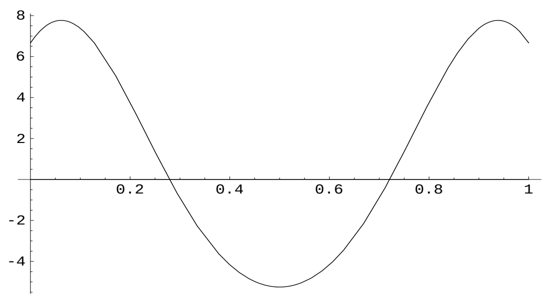
whereas if $n \geq 1$ is even then

$$d\alpha_n(v_n, [\zeta_n, v_n]) = -2\Big\{(\widetilde{A}_n + \widetilde{B}_n)\frac{h_n}{R_n} + 2n\pi a^2 r_1 r_2 K_n\Big\} =: -2Q_n$$

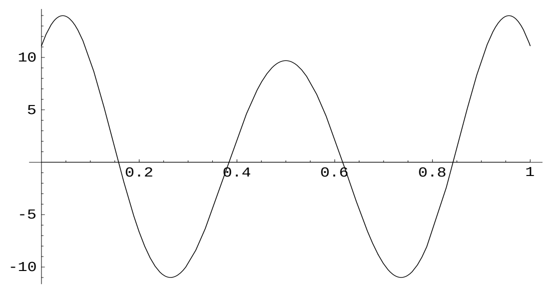
where

$$K_n := \widetilde{A}_n(n\pi r_2 B_n - 2A_n) + \widetilde{B}_n(n\pi r_1 A_n - 2B_n), \qquad a = \frac{1}{\sqrt{r_1 r_2}}(x_1x_3 - x_2x_4)$$

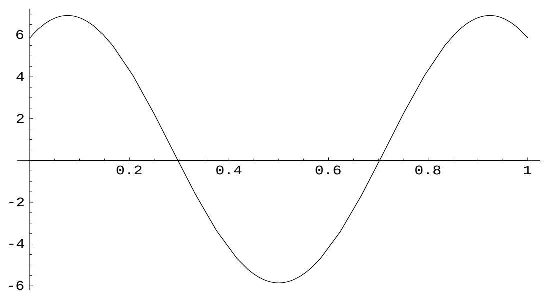
Now, for every $0 < r_1 < 1$, there exist x_1, x_2, x_3, x_4 (even along the periodic orbits of ξ_n) such that $a^2 = 0$. In such a points the sign of $d\alpha_n(v_n, [\zeta_n, v_n])$ depends on $\widetilde{A}_n \pm \widetilde{B}_n$ and we find the following graphs Thus in general the hypothesis (ii) does



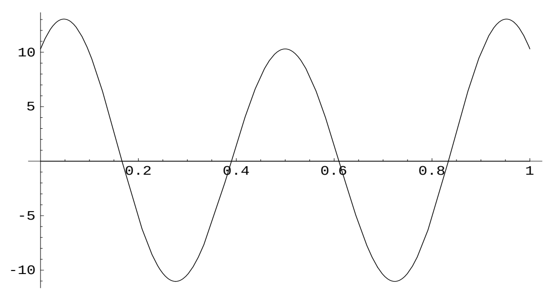
(g) $\widetilde{A}_n - \widetilde{B}_n, n = 3$



(h) $\widetilde{A}_n - \widetilde{B}_n, n = 5$



(i) $\widetilde{A}_n + \widetilde{B}_n, n = 2$



(j) $\widetilde{A}_n + \widetilde{B}_n, n = 4$

not hold in general. Anyway the existence of a "good" v_n (for which the hypotheses $(i), (ii)$ are satisfied) is not a priori excluded.

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